THE CONIC SECTIONS

CONSIDERED IN REFERENCE TO THEIR PRACTICAL APPLICATION.

The Parabola.

The *Parabola*, according to the definition given at page 462 of the current volume, is formed by a plane passing through a cone in a direction parallel to the slant side thereof; it is therefore a curve of such a nature, that if any number of points be assumed in the axis, and through these points perpendiculars be drawn to meet the curve in either direction:—

The distances between the vertex or origin of the axis and the several assumed points, are respectively proportional to the squares of the corresponding perpendiculars.—Or, in other words, *the abscisses are proportional to the squares of their corresponding ordinates.*

From this proposition the general equation or characteristic of the curve is derived, and the manner of its derivation may be illustrated as follows:—

Let ABC, fig. 2, be a section along the axis of a right cone, of which C is the vertex and AB the diameter of the base, and let DVE be a parabolic section, made by a plane passing through the cone in a direction perpendicular to the plane ACB, and parallel to the slant side CB; then is VF the line of common intersection of the two planes ACB, DVE, and DE the line of common intersection of the cutting plane DVE with the base of the cone, so that V is the vertex of the parabolic section, VF its axis parallel to CB, and DE its base, FD and FE being ordinates to the axis in the point F.

Through any point I in the axis VF, and parallel to the base DE, draw the straight line GH meeting the curve both ways in the points G and H; then are IG and IH ordinates to the axis in the point I, GH being a double ordinate corresponding to DE, the base of the section.

Through the same point I and parallel to AB, the base of the cone, draw the straight line KL, meeting the slant sides CA and CB in the points K and L; then is KL the diameter of the circular section through the point I, any how assumed in VF, the axis of the parabola.

Now, since the ordinates FD and IG meet the circumferences of the circular sections ADB and KGL, as well as the parabolic curve DVE in the points D and G, they are respectively perpendicular to the diameters AB and KL, and, consequently, they are ordinates to these diameters, as well as to VF the axis of the parabola; for because the planes ACB and DVE are perpendicular to one another, it follows from the principles of solid geometry, that DE and GH are also perpendicular to AB and KL. From this construction, therefore, we are to prove, that, *the abscissa* VI, *is to the abscissa* VF, *as the square of the ordinate* IG, *is to the square of the ordinate* FD; because by construction the straight line KI is parallel to AF, the triangles KVI and AVF are similar to one another, and, consequently, by the property of similar triangles, we have

$$VI : VF :: KI : AF;$$

but by the nature of proportion, if the consequents of an analogy be both multiplied by the same quantity, the ratio is not altered in consequence of such multiplication.

Now, since VF the axis of the parabola, is parallel to CB, the side of the cone, and KL parallel to AB, it follows that IL and FB are equal to one another, being opposite sides of the parallelogram ILBF; hence it is

$$VI : VF :: KI \times IL : AF \times FB;$$

but by the property of the circle, the rectangles or products KI × IL and AF × FB, are respectively equal to the squares of the ordinates IG and FD; therefore, by substitution, we get as follows,

$VI : VF :: \overline{IG}^2 : \overline{FD}^2$, which was to be proved.

If this analogy be converted into an equation, by making the product of the mean terms equal to the product of the extremes, one form of the equation to the parabolic curve becomes

$$VI \times \overline{FD}^2 = VF \times \overline{IG}^2 \ldots\ldots(A.)$$

Now, in this equation, if any three of the quantities be given, the fourth can always be found, by simply disengaging the required quantity from that with which it is combined, by division and the extraction of roots according to the combination.

If it were required to determine the ordinate IG from the abscissæ VI, VF, and the ordinate FD; we have only to divide both sides of the equation by VF, and extract the square root of the quotient, and we get $IG = FD \sqrt{\frac{VI}{VF}}$. And in like manner, each of the other quantities in terms of the rest may be expressed as follows, viz.:—abscissa $VI = \frac{VF \times \overline{IG}^2}{\overline{FD}^2}$; abscissa $VF = \frac{VI \times \overline{FD}^2}{\overline{IG}^2}$; and finally, ordinate $FD = IG \sqrt{\frac{VF}{VI}}$.

From these equations the following practical rules are deduced, according as it is an ordinate or abscissa that is required; when it is an ordinate, as in the first and fourth of the above equations, the rule is as follows;—

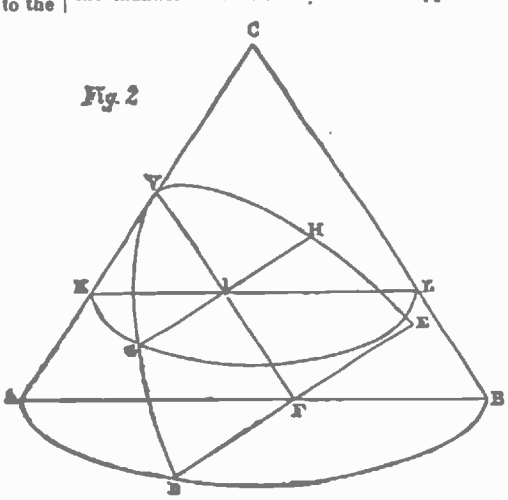
RULE.—*Divide the abscissa belonging to the required ordinate by the other abscissa belonging to the given ordinate; multiply the square root of the quotient by the given ordinate, and the product will be the ordinate required.*

This rule for the ordinate is very concise and easy of application; but when an abscissa is required as in the second or third of the above equations, the rule is as follows:—

RULE.—*Multiply the square of the ordinate belonging to the required abscissa by the given abscissa, and divide the product by the square of the other given ordinate for the abscissa required.*

By these two rules all questions respecting the ordinates and abscissæ of the axis can be resolved, and the following examples will shew the manner in which they are to be applied.

Fig. 2



Example 1. If the ordinate corresponding to an abscissa of 48 inches be 18 inches, what is the ordinate corresponding to an abscissa of 14 inches?

By a reference to the figure, it will readily appear, that we have given the abscissæ VI, VF and the ordinate DF, to find the ordinate IG, corresponding to the lesser abscissa VI. Therefore, by the first of the above rules, it is $\sqrt{\frac{14}{48}} = 0.54006$; and $0.54006 \times 18 = 9.7211$ in.

Example 2. If the abscissa corresponding to an ordinate of 12 inches, be 17 inches, what is the abscissa corresponding to an ordinate of 27 inches?

Here again, by reference to the figure, we find that the given quantities are VI, VF, and IG, to find FD the ordinate corresponding to the greater abscissa VF; and for this purpose, the second of the above rules gives the following process:—Square of given ordinate, $27 \times 27 = 729$; square of the other ordinate, $12 \times 12 = 144$; then $729 \times 17 = 12393$; and $\frac{12393}{144} = 86\frac{9}{16}$ inches.

And in precisely the same manner are the rules to be applied to any other example, taking particular care to apply the ordinate or abscissa given, corresponding to the abscissa or ordinate required exactly as directed, otherwise the process will lead to a very false result. This is all the difficulty that can occur in the application of the rules, but a very little attention and practice will be sufficient to guard against the liability to fall into error on this point.

If we return to the analogy from which the general equation (A) was derived, and divide each consequent by its antecedent, we shall find that the expression $\frac{\overline{IG}^2}{VI}$, or $\frac{\overline{FD}^2}{VF}$ is a constant quantity for the same parabola, to whatever point of the axis the quantities may be referred; but as each of these terms expresses a ratio by the nature of proportion, they may be converted into an analogy, as follows:—

$VI : IG :: IG : \frac{\overline{IG}^2}{VI}$; and $VF : FD :: FD : \frac{\overline{FD}^2}{VF}$

from which it appears that in each case, the fourth term of the analogy is a third proportional to the abscissa of the axis and its corresponding ordinate; but by the definitions page 463, the third proportional to an abscissa and its corresponding ordinate is equal to the parameter of the axis, or to the double ordinate which passes through the focus. Let this element be denoted by the symbol p, and let it be substituted for the terms $\frac{\overline{IG}^2}{VI}$ and $\frac{\overline{FD}^2}{VF}$ in the above proportions, and they become

$VI : IG :: IG : p$; and $VF : FD :: FD : p$.

Let each of these analogies be converted into an equation, by making the product of the mean terms equal to the product of the extremes, and we shall have

$$p \times VI = \overline{IG}^2; \text{ and } p \times VF = \overline{FD}^2;$$

so that generally, to whatever point of the axis the ordinate may belong, its square is always equal to the rectangle of the corresponding abscissa drawn into the constant quantity p. Therefore, if x be put to denote any abscissa estimated from the vertex, and y the corresponding ordinate; then the general characteristic of the curve, on which its several properties are dependent, becomes

$$p x = y^2 \ldots\ldots\ldots(B.)$$

But for the convenience of practical men, we shall express this equation in a specific form, as follows:—

Parameter × *abscissa* = *ordinate* × *ordinate*.....(C.)

Now this is a very simple and elegant expression for a curve of such extensive application in practice as the parabola is, and the rule which it supplies is thus enunciated.

RULE.—*Multiply the given abscissa by the parameter, and extract the square root of the product for the ordinate required.*

Example 3. What is the ordinate corresponding to an abscissa of 24 inches, supposing the parameter of the axis to be 6 inches?

Here, according to the rule, we have $24 \times 6 = 144$, the ordinate squared; therefore, by taking the square root of 144, we get 12 for the ordinate required.

From the same general equation (C), we may determine the abscissa corresponding to any given ordinate, when the parameter is known; for we have only to divide both sides of the equation by the parameter, and the quotient will be the abscissa sought; thus we have

abscissa = *ordinate* × *ordinate* ÷ *parameter*.....(D.)

The practical rule derived from this equation is expressed in words, as follows:—

RULE.—*Divide the square of the given ordinate by the parameter of the axis, and the quotient will be the abscissa sought.*

Example 4. What is the abscissa corresponding to an ordinate of 12 inches, supposing the parameter of the axis to be 6 inches?

Here, by operating according to the rule, we have $12 \times 12 = 144$; and $144 \div 6 = 24$ inches, the abscissa required.

It frequently happens, however, in the application of the parabola, that the parameter, or constant quantity is not given; indeed, in almost every case where the curve is to be delineated, the parameter requires to be found; but in every instance such data must be proposed as will enable us to determine what that element is, for we must either have given two abscissæ and one ordinate, or two ordinates and one abscissa, and from an ordinate

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